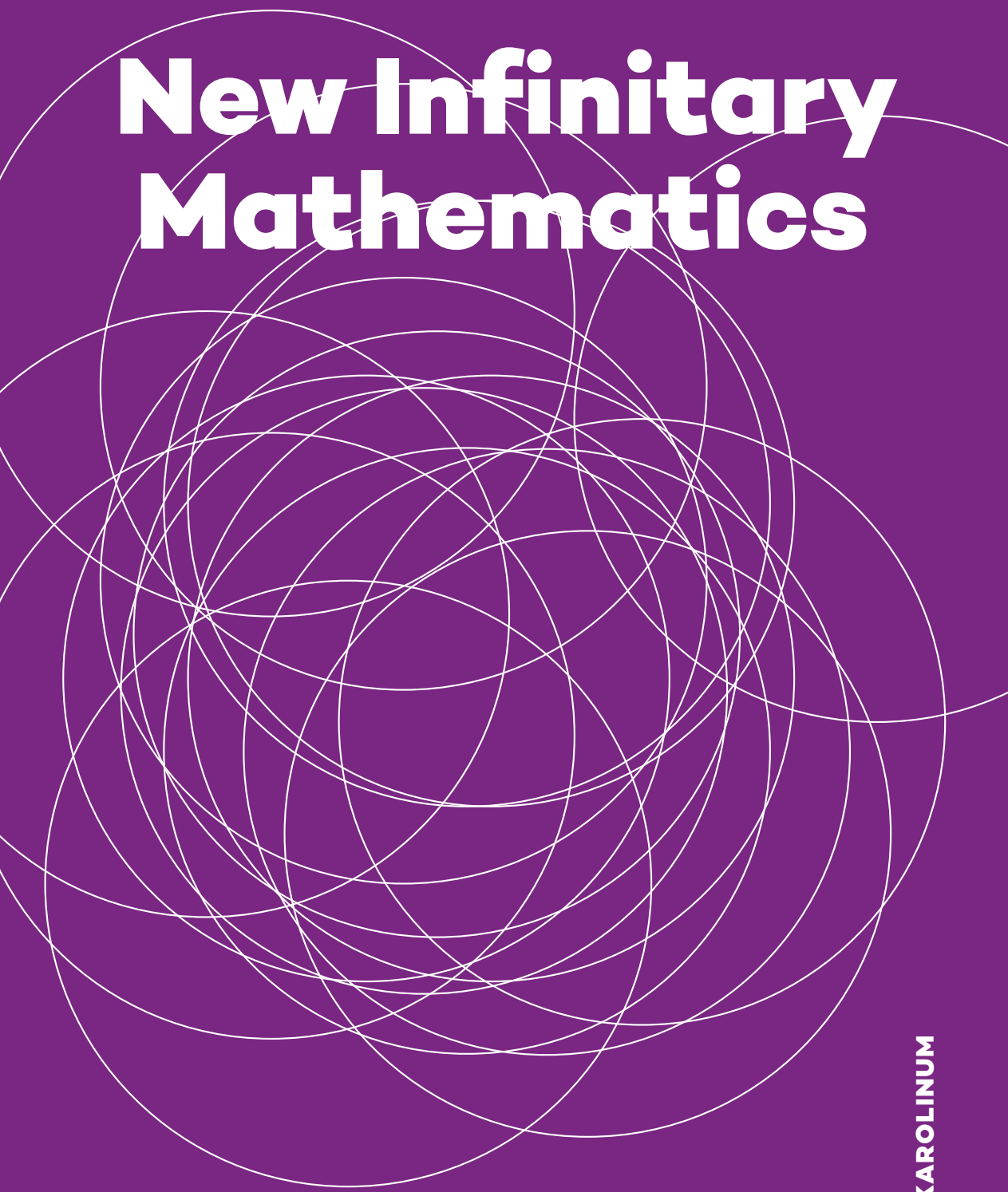


**Petr Vopěnka**

# **New Infinitary Mathematics**



**KAROLINUM**

# New Infinitary Mathematics

Petr Vopěnka

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## Editor's Note

The original reason for this book was the consensus that Vopěnka's mathematical and philosophical contributions made after he left mainstream set theory should be available in English. Bringing the book to publication has taken ten years for the following reasons: first Vopěnka wrote another manuscript in Czech<sup>1</sup> subsequently translated by Hana Moravcová and Roland Andrew Letham, called *The Great Illusion of the Twentieth Century Mathematics*. However, it turned out that the translation of some parts of the text needed more relevant mathematical expertise and Alena Vencovská took on the task of making it correct. The author used the opportunity to extend and modify the book considerably. He worked on it until his sudden death in 2015. The result was twofold: more publications in Czech, namely the four-volumed work *New Infinitary Mathematics*,<sup>2</sup> along with *Prolegomena to the New Infinitary Mathematics*,<sup>3</sup> and a parallel English text with additions to the original book translated by Vencovská. The Czech and English versions differed little from each other, except that the order of the material was different, and Vopěnka left some parts out from the English version. In particular, he did not include what are now the first two chapters, and some sections throughout. This present version does include these initial chapters (on the theological foundations of Cantor's set theory and on its rise and growth, the former translated by Václav Paris) but it does not include all that is in the Czech version.

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<sup>1</sup> Petr Vopěnka, *Velká iluze Matematiky XX. století a nové základy* (Plzeň: Západočeská univerzita v Plzni a Nakladatelství Koniáš, 2011).

<sup>2</sup> Petr Vopěnka, *Nová infinitní matematika* (Praha: Karolinum, 2015).

<sup>3</sup> Petr Vopěnka, *Prolegomena k nové infinitní matematice* (Praha: Karolinum, 2013).



## Editor's Introduction

### About the Author

Petr Vopěnka grew up in the former Czechoslovakia, where he was born in 1935 (to parents who both taught mathematics at a secondary school). He enjoyed scouting in his youth and often remembered times spent at camps. In a way he remained true to the values he formed early on all through his life. Personal integrity, faith in truth prevailing over deceit, loyalty to friends, great love for his troubled country and an unshakeable commitment to his work were some of his most striking characteristics. To this, one needs to add that he loved to laugh.

For much of his life, Czechoslovakia was ruled by the communists: they took over in 1948, and education during Vopěnka's teenage years bore the stamp of Stalinism. Vopěnka reminisced about being asked to take turns in a whole day of reading funeral poems on the school radio upon Stalin's death in 1953, and he arrived in Prague later the same year to study mathematics in a city overlooked from a hill by Stalin's 16-meter-high statue. Fortunately, mathematics is relatively immune to ideological manipulation and Vopěnka remembered his student years and his teachers fondly.

His early research was mainly in topology and he wrote his master's thesis under the supervision of Eduard Čech, an eminent topologist and geometer, whose name lives for example in Čech cohomology and Čech-Stone compactification. Vopěnka used to say that Čech "showed him how to do mathematics". The research that he engaged in at that time concerned compact Hausdorff spaces and their dimensions.

Soon after graduation, Vopěnka started to teach mathematics at Charles University and he remained there for most of his professional life. Quite early on, he developed an interest in mathematical logic, championed in Czechoslovakia by Ladislav Rieger who wrote about the subject for the Czech mathematical community and ran a seminar on set theory. Vopěnka participated and, after Rieger's untimely death in 1962, took over as its organiser to provide strong and inspired leadership for Czechoslovak mathematical logicians. Vopěnka published work on nonstandard interpretations of Gödel-Bernays set theory based on using the ultrapower construction and then in collaboration with the seminar participants he contributed substantially to the exciting discoveries following Gödel and Cohen's groundbreaking work on the consistency and independence of the continuum hypothesis and the axiom of choice. Due to the Iron Curtain, communication with other mathematicians working in the area was limited and some results obtained independently in Prague came later than those in the West, but others remain credited to the Prague group. By all accounts it was as vibrant and fruitful a period as can be—Alfred Tarski wrote about the community in these words:<sup>4</sup> "I do not know if there is at this point another place in the world, having as numerous and cooperative a group

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<sup>4</sup> Quoted in Antonín Sochor, "Petr Vopěnka (born 16. 5. 1935)," *Ann. Pure Appl. Logic* 109 (2001): 1–8.

of young and talented researchers in the foundations of mathematics.”

This lasted some years, but then two factors caused it to fall apart. One was that Vopěnka became very sceptical about the role that set theory, as it was, could have in truly explaining the phenomenon of infinity and in serving as a foundation for mathematics. It mattered to him; he did not wish to explore that intricate and bewitching maze any further so he started to look for alternatives. Paradoxically, the one concept which is today perhaps most strongly associated with Vopěnka within this area arose as he was abandoning the subject, when he proposed what became to be known as Vopěnka's principle. This yields a strong large-cardinal axiom that Vopěnka said he believed he could prove to be contradictory, suggesting it merely to make the point that investigating consequences of more and more set-theoretical axioms made little sense. However, Vopěnka's argument that it was contradictory contained an error, and interest in the axiom prospered outside of Czechoslovakia. Tightening controls within the country again limited communication with the West for academics like Vopěnka so it was some years later that he learnt with surprise that this principle was still alive and well established.

The other factor that contributed to the demise of this golden era of mainstream set theory in Prague were the political events—the 1960s brought a gradual thaw of orthodox communism leading to Prague Spring in 1968. This however was followed by the August 1968 invasion whereby the Warsaw Pact armies put an end to it. Some of Vopěnka's collaborators, in particular Tomáš Jech and Karel Hrbáček left the country, and most of the others sought their own independent paths. Vopěnka, who prior to 1968 had joined the efforts led by Alexander Dubček to reform communism and had gained some influence in running the Faculty of Mathematics and Physics at his university, would not support the official line after the invasion and might well have been forced to leave the university along with many other academics in similar positions. He was allowed to stay to do research, although his contact with students was very restricted. Many years later when he learnt that he owed this good fortune to the intervention of the Soviet mathematician P. S. Alexandrov, he used to joke that had he known how powerful a protector he had, he would have been braver (standing up to the suffocating pressure of the Czechoslovak communist “normalisation” of the 1970s and 1980s). In fact, he was one of the few who did stand up to it in any way that seemed possible.

At this turning point, Petr Vopěnka along with Petr Hájek wrote a book on semisets,<sup>5</sup> exploring set theories obtained by modifying the usual von Neumann-Bernays-Gödel axioms for classes and sets so that sets can have subclasses that are not themselves sets (proper semisets). Apart from the importance of semisets for forcing, Vopěnka's new motivation was investigating other ways in which the phenomenon of infinity could be captured mathematically, better reflecting how we encounter infinity when thinking about the world, often as a part of a large finite set. The book did not dwell on this aspect though and

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<sup>5</sup> Petr Vopěnka and Petr Hájek, *The Theory of Semisets* (Prague: North Holland and Academia, 1972).

focused on providing a careful formal development of the theory of semisets and on showing its suitability for finding models of set theory via forcing.

Vopěnka then moved on to formulate a different set theory, which he hoped would capture his intuition about infinity in a better way. It was an intuition gained through much reflection on what we understand by infinity and how we see the world, influenced mainly by Bolzano and Cantor's writings, by discussions surrounding the birth of set theory and by the philosophy of Husserl and Heidegger (for many years there was a weekly seminar taking place in Vopěnka's office devoted to the study of their work). Semisets were a step in the right direction, but Vopěnka wished to formulate a new theory from the position of a mathematician free of any commitment to the current view of infinity; to develop mathematics as it might have been developed if satisfactory axioms for infinitesimals had been found before mathematics took its present course.

This led to what he called the alternative set theory. It contains sets and classes; sets alone behave as classical finite sets but they may contain subclasses which are not sets (semisets). Unlike Cantorian set theories, alternative set theory admits only two types of infinity: the countable infinite and the continuum. This is not a necessary requirement of such a set theory, it could be constructed otherwise, but Vopěnka's motivation was to keep only what could be justified by some intuition other than intuition arising purely from the study of set theory itself; for him it meant just the infinities associated with natural numbers or with the real line. A crucial principle in Vopěnka's alternative set theory is the Axiom of Prolongation, related to the phenomenon of the horizon (understood in a very general sense). It reflects the intuition that something seen to behave in a certain well-defined way as far as the horizon will continue to do so beyond the horizon.

Mathematically, the theory is close to the concept of nonstandard models of natural numbers underlying nonstandard analysis. However, from a foundational point of view there is a considerable difference since in nonstandard analysis infinitesimals are complicated infinitary objects whilst in AST some exist just as rational numbers do. Formulating a theory that allows mathematical analysis to be practiced in a way in which it was conceived by Leibniz, that is as a calculus with infinitesimals, was indeed one of Vopěnka's objectives. This had not been done within the alternative set theory at the time, and Vopěnka returned to the task in this book.

Vopěnka succeeded in assembling another group of enthusiastic mathematicians, who wanted to work with him and develop AST. One unfailingly supportive and faithful collaborator from before also joined him in the endeavour, Antonín Sochor. Interesting results were obtained, first within the Prague circle and later on also at other places in the world, but overall its impact was relatively small. In particular, investigations of alternative set-theoretical universes was restricted to what Vopěnka called a limit universe (as opposed to a witnessed universe). In a limit universe no "concrete" set such as the set of natural numbers less than  $67^{293^{159}}$  can contain semisets but in a witnessed universe some can. The witnessed universes correspond to Vopěnka's intuition,